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ORIGINAL PAPER

The strange role of calculus in the United States

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Abstract
In the United States, the first course in single-variable calculus is considered tertiary level mathematics. Initially offered in high schools as a means for strong students to do college-level work, it is now taken by over 20% of high school students and perceived to be a prerequisite for admission into selective colleges and universities. This article describes the growth of this phenomenon and its effects on issues of educational equity. Because U.S. schools are funded locally, there is tremendous variation in the availability of calculus instruction in high school, with the most privileged students having the greatest access. This has profound effects on issues of equity because few universities are effectively addressing the vast disparities in student preparation. This article concludes with observations on what can and should be done to ameliorate the strange situation in the United States with regard to calculus.

Keywords Mathematics education · Calculus · Advanced placement calculus · Equity · MAA Committee on the Undergraduate Program in Mathematics

1 Introduction

In the United States, the first course in single-variable calculus is considered tertiary level mathematics. In the 1950s the College Board's Advanced Placement program began offering this university course to increasing numbers of high school student with the intention of allowing mathematically strong students to do university level work while still in high school. Over the past several decades, calculus offered in high school has grown to the point where it is now perceived to be a prerequisite for admission into selective colleges and universities. Today it is taken by over 20% of high school students. At the same time, universities have not adapted to this growth, still teaching their first calculus course as if students had never before seen the material.

To appreciate the issues surrounding calculus instruction in the United States, it is important to understand some of the idiosyncrasies of its public school system and its tertiary education. In the U.S., most public school funding is raised within the community in which the schools are located. Because of this, decisions on funding levels, curricula, and staffing are made locally. This means that there is tremendous disparity across the country in what courses are offered and how teachers are prepared to teach these courses. This is especially true for calculus in high school. Because of the lack of uniformity in access to calculus in high school and widespread reluctance by universities[1] to recognize calculus learned in high school, most college Calculus I classes in the United States contain students who are completely new to the terminology and concepts of calculus and students who have already demonstrated proficiency in all of the topics to be covered in that course.

The uniqueness of the system in the U.S. goes beyond problems of articulation between high school and college. In most U.S. universities, students are not required to declare their major until the end of their second year, and most choices of major are set up to allow students as much flexibility as possible to switch majors. The result is that there is usually one "mainstream" calculus sequence that serves as a prerequisite for further mathematical studies regardless of major. It is taken by students pursuing degrees in mathematics, statistics, computer science, physical sciences, biological sciences, social sciences, education, and engineering

[1] This article uses the terms "college" and "university" interchangeably. Universities have graduate programs; college do not. But the United States has an extensive system of colleges, many of which are as selective as the very selective universities. While universities tend to teach Calculus in larger classes, there is very little difference between colleges and universities in the syllabus or the expectations for this course.

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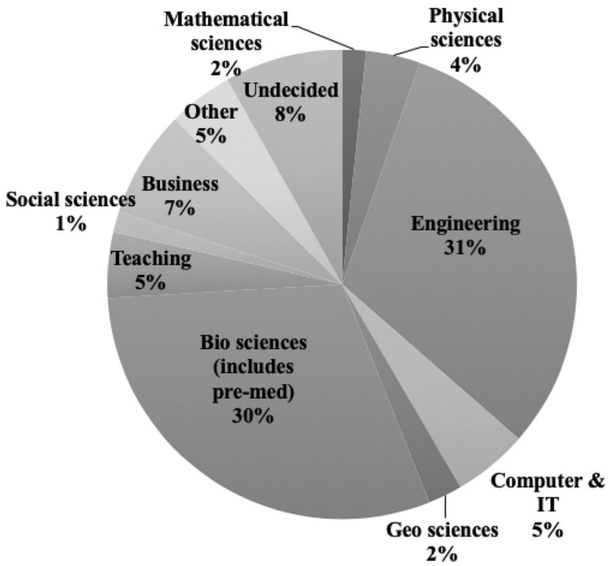

Fig. 1 Distribution of intended majors of all students in mainstream Calculus I. N = 6984. Source: Bressoud, 2017

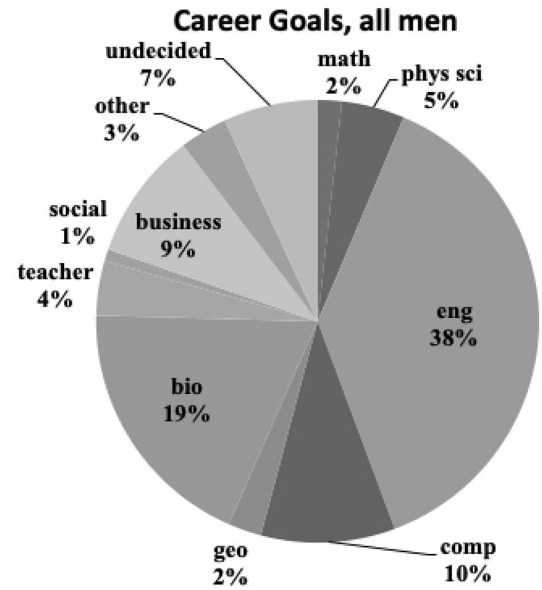

Fig. 2 Distribution of intended majors of all male students in mainstream Calculus I. N = 3816. Source: Bressoud, 2017

(Fig. 1).[2] Thus, the same course is trying to serve those who will need to build on it for further mathematical studies and students who see this as the last mathematics course they will take. As an illustration of the disparities within this single course, the Mathematical Association of America (MAA) found that a majority of the men in Calculus I were majoring in engineering or a physical science, while a majority of the women were majoring in the biological sciences or education (Figs. 2 and 3) (Bressoud, 2015).

The result is a broad course designed for students across a wide variety of abilities and interests. The emphasis is on procedures and common applications beginning with differential calculus. The course usually introduces indefinite integrals before moving quickly through the Riemann definition of the definite integral on its way to techniques of integration and applications. The first year of calculus typically concludes with a discussion of infinite series, especially Taylor series, and tests of convergence.

Because of the uniformity of mainstream Calculus, the College Board's Advanced Placement (AP) program has been able to craft a high school course that corresponds to what is taught in almost all colleges and universities.

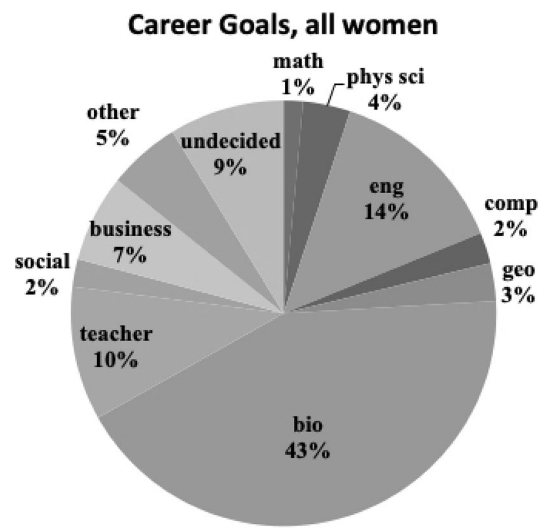

Fig. 3 Distribution of intended majors of all female students in mainstream Calculus I. N = 3137. Source: Bressoud, 2017

The original intention of the AP Calculus program was to enable mathematically strong students to take this course while in high school and so proceed directly on to more advanced mathematics when they arrived at university. As we will see, this program still fulfills this purpose, but to a minority of the students who enroll in it. This article will explain the phenomenon and impact of AP Calculus beginning with a description of its historical development and growth, describing the factors that have fostered this

[2] Most universities offer a separate calculus sequence for business majors that focuses on basic concepts applied primarily to polynomial functions. While many biology majors take mainstream calculus, their calculus experience is highly varied across institutions. It ranges from no calculus required for the major through courses that are more like business calculus to what is effectively mainstream calculus with an emphasis on biological examples. The calculus described in this article is only mainstream calculus.

growth. We will examine the concerns this has raised within the mathematical community.

We will then turn to studies of this program, first those sponsored by the College Board to examine the validity of the AP Calculus Exams as a measure of readiness for advanced placement, then the Sadler and Sonnert study begun in 2009 that examined the factors, including high school calculus, that indicate preparedness for college-level Calculus I. This article will discuss the impact of high school calculus on issues of equity and conclude with observations of what can and should happen to ameliorate the current situation.

2 The rise of AP calculus

The AP program arose from efforts in the early 1950s to equalize opportunities for talented students to do college-level work before graduating from high school. The program established national examinations in key subjects that, irrespective of the high school at which the student had taken this course, would provide recognition that a college requirement had been satisfied.

This was at a time when the expectations for the first year of university mathematics were still in flux, even for prospective mathematics majors. By 1956, the decision had been made that the emphasis for the mathematics examination would be on calculus. The oversight of this program was handed over to the College Board, a private not-for-profit organization that already had 30 years of experience in the administration of what was then known as the Scholastic Aptitude Test (SAT) (Bressoud, 2010; Cornog, 1957; SCS, 1956).

The impact of AP Calculus was small at first. The College Board did not administer one thousand calculus exams per year until 1960, and did not reach 10,000 per year until 1967. In 1973, the number of exams was still just over 14,000. But then two things happened to greatly accelerate its growth. One was the decision, starting in 1969, to allow students to be tested just on the first semester of single variable calculus, the "AB Exam," allowing many more students to aspire to some college credit for calculus. The full year of college calculus would still be assessed as the "BC Exam." In 1981, the number of AP Calculus exams passed 30,000 per year for the first time.

Then, in 1982, the story of Jaime Escalante and his success at Garfield High School in East Los Angeles broke (Matthews, 1988). He demonstrated that if you give students challenging material—with the right supports—they can achieve far more than traditionally expected. In 1984, Governor Richard Riley of South Carolina, later President Clinton's Secretary of Education, sought to improve the performance of his state's schools by mandating the inclusion of AP programs in every high school (Anderson, 2016). Many other states followed. In 1993, over 100,000 AP Calculus exams were administered.

When the effort to reform calculus instruction emerged in the late 1980s, peaking from 1989 to 1995, many of those at the head of this movement were also in senior positions within AP Calculus. They shaped this program to embrace graphical and numerical as well as the traditional algebraic representations of functions. They greatly expanded the need to justify solutions, requiring citation of the relevant theorem and verification that the needed assumptions were satisfied. And they made the use of technology an integral part of the course. This has turned into one of the more controversial aspects of AP Calculus. Because of the need to choose a technology that is accessible to all students, the graphing calculator was selected. This has remained the technology of choice because access to computers is still limited in many high schools and graphing calculators, which do not have access to the internet, can be used during examinations. Driven by their role in high school calculus, graphing calculators migrated throughout the high school curriculum. This 30-year-old technology still dominates high school mathematics in the United States.

Today, most high school students use a calculator equipped with a Computer Algebra System (CAS). While these calculators are only permitted on a portion of the AP Calculus exam, this creates disjunction when students enter university. Because there is such a range of attitudes toward the use of technology within post-secondary calculus, both ends of the spectrum cause difficulties for students: frustration with the inability to access CAS when taking a college course that does not permit the use of such technology, discontent with the limitations of their high school technology when taking a college course that embraces the use of computers.

By the early 1990s, a self-reinforcing dynamic had set in. Most of the students taking the AP Calculus exam, regardless of how well they did, were advised or simply preferred to retake Calculus I in college. Word got back to the high schools that students who had not seen calculus before they arrived at university were competing against those who had already passed the course in high school. From a 1993 survey, Bressoud (1994) observed that many students in Calculus I at the Pennsylvania State University believed that success in calculus in high school was necessary for success in calculus in college. The nationwide survey conducted by the Mathematical Association of America (MAA) in 2010 revealed that 46% of the students agreed with the statement, "In order to succeed in calculus at a college or university, I must have taken it before." (CSPCC, 2011) The perception continued to grow that one faced a severe disadvantage in college calculus if it had not been taken in high school (Seymour and Hunter, 2019). Enormous pressures built,

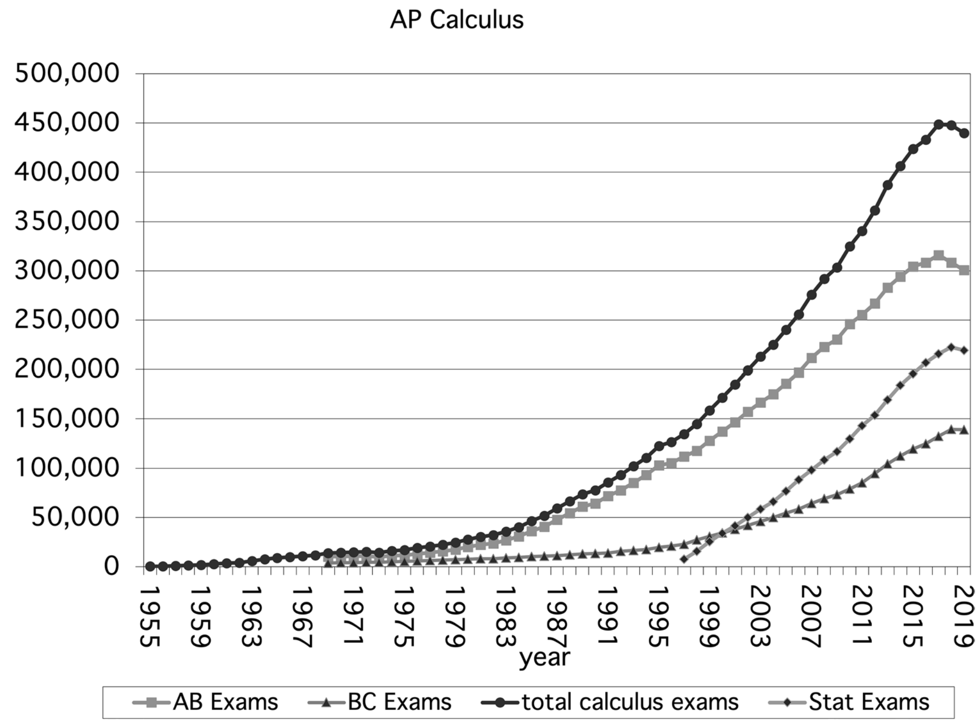

Fig. 4 Number of students who took AP exams in mathematics: Calculus AB, Calculus BC, and Statistics. Source: research.collegeboard.org/programs/ap/data and personal communication from AP Calculus Chief Reader

especially in well-resourced school districts, to make AP Calculus available to as many students as possible. Exam administrations passed 200,000 per year in 2003, 300,000 in 2009, and 400,000 in 2014.

Comparing the number of students who took an AP Calculus exam in 2013 with the number of those 2013 high school graduates with calculus on their transcript (Champion and Mesa, 2017), we can estimate that about 55% of the students who study calculus while in high school take the exam. While the number of students taking the AP Calculus exam seems to be leveling off at around 450,000 per year (College Board, 2019a), the growth in the program since 2013 suggests that today high school calculus in one form or another is taught to around 800,000 students, over 20% of all high school graduates (Fig. 4).

3 Pressure to enroll in high school calculus

Today high school enrollment in calculus is widely perceived to be a necessary indication that one has the ability to succeed in the most prestigious and selective universities. As Dan Teague, a teacher at the North Carolina School of Science and Mathematics, has stated, “The goals of most high school students taking calculus are focused on college admissions rather than mastery of the content” (Teague, 2017, p. 45). By 2010, two-thirds of the students in the first mainstream college calculus course were repeating a class they had taken in high school (Bressoud, 2017).

In 2011, Rosenstein and Ahluwalia (2017) surveyed the 478 graduating seniors who had entered Rutgers University with an AP Calculus score on their transcript, seeking to understand why they had chosen to take this course while in high school. Of these, 194 students responded. They were offered fifteen possible reasons. The responses were separated by whether the student had entered with a 4 or 5 on the BC exam, equivalent to an A or a B in the course, (37 students, 19%), a 4 or 5 on the AB exam (58 students, 30%), or 3 or lower on the exam they took (99 students, 51%). The top five responses from each group are informative.

Those who earned a 4 or 5 on either exam agreed on four of the five responses:

- I really liked math when I was in high school,
- I wanted to learn more higher level mathematics,
- I enjoy challenging math courses,
- I wanted to start off college with a higher level course.

Those with a 4 or 5 on the BC exam added, “My friends were taking AP Calculus.” Those with a 4 or 5 on the AB exam added “AP Calculus looks good on college applications.”

Those who had earned a 3 or lower had very different responses. They agreed with the other groups that “I wanted to start off in college with a higher level course” and with those earning a 4 or 5 on the AB exam that “AP Calculus looks good on college applications.” They added three very different rationales:

- I wanted to be better prepared for college courses,
- My math teachers or counselors suggested that I should take it,
- I had to take some math course in my senior year.

Rough estimates (Bressoud, 2017; Christman Morgan, 2002) suggest that just over half the students who study calculus in high school will take a mainstream calculus class in college. About a fifth of those who have studied calculus in high school earn and take advantage of advanced placement, starting in college with Calculus II or higher. About a third will retake Calculus I.

Calculus as taught in U.S. high schools appears to be serving three different roles. It does fulfill the original vision of offering an option for advanced placement for a talented group of students who enjoy mathematics and want to go as far with it as they can, that top fifth. The middle group see it primarily as preparation for a course of study that will require calculus. The remainder take it as a course that will help their chances of admittance to their desired college or university.

With the exception of a few highly selective technical programs, no college or university in the United States requires prior study of calculus for admission. But admissions officers do look for evidence that the applicant has the ability to succeed at their institution. Successful completion of calculus in high school is a positive signal. In the competitive process of college application, families that are aware of this advantage will strive to enroll their children in this course.

Improving the chances of acceptance into the university of one's choice is part of the rationale for those who will not continue with calculus, but the last bullet point, "I had to take some math course in my senior year," also plays a large role. The standard 4-year high school curriculum is geared toward preparing students for calculus: Algebra I, Geometry, Algebra II, Precalculus. Increasingly, this progression has shifted into earlier grades. Especially for students who show any mathematical ability in middle school and for those from privileged schools, Algebra I starts in grade 8 or even grade 7. Calculus becomes the default course once Algebra II and Precalculus have been passed.

4 Pushing back on calculus in high school

In 1983 the MAA's Committee on the Undergraduate Program in Mathematics (CUPM), alarmed that over 200,000 high school students were now studying calculus, established the Calculus Articulation Panel to understand and make recommendations relevant to this phenomenon. The Panel raised five concerns (CUPM, 1987):

1. Qualifications and expectations of those teaching calculus in high school,
2. Qualifications and expectations of the students taking calculus in high school,
3. The effect of repeating a course in college that had already been passed in high school,
4. How students were to be placed when they got to college,
5. Lack of communication between high schools and colleges.

The Panel made a number of recommendations that were summarized in a 1986 letter issued jointly by the Presidents of MAA and the National Council of Teachers of Mathematics (NCTM), Lynn Steen and John Dossey (Steen & Dossey, 1986). They made two points:

1. "The calculus course offered in the 12th grade should be treated as a college-level course." It should be taken with the expectation that it would enable the student to take advantage of advanced placement, meaning that graduates should expect to be able to place directly into the next calculus class at university rather than "as an introduction to calculus with the expectation of repeating the material in college." To ensure this, students taking calculus in high school should be expected to either take the AP Exam or a comparable examination capable of conferring college credit.
2. Before enrolling in calculus, "students should have at least four full years of mathematical preparation beginning with the first course in algebra." This includes "algebra, trigonometry, analytic geometry, complex numbers, and elementary functions studied in depth." The presidents specifically warned against studying calculus in high school if the study of algebra had not begun by grade 8.

The CUPM Panel had made a total of 17 recommendations. Perhaps the most important that were not reflected in the presidents' letter were two that were directed at the universities, numbers 15 and 16:

> 15. "Colleges should develop special courses in calculus for students who have been successful in accelerated programs, but have clearly not earned advanced placement."
> 16. "Colleges should establish periodic meetings where high school and college teachers can discuss expectations, requirements, and student performance."

Few universities acted on either of these recommendations. For the past several decades, the AP Calculus program has provided one of the few venues where mathematicians and high school teachers regularly meet. Almost

Table 1 Comparison of students who used AP exam grade to skip a course versus students who took and passed that course. Source: Morgan & Ramist, 1998

Placed via	Average grade in Calculus II	N
Passed Calculus I	2.52	11,212
3 on AB exam	2.67	877
4 on AB exam	2.79	1159
5 on AB exam	3.23	1078
Passed Calculus I	2.51	10,366
3 on BC exam	2.88	333
4 on BC exam	3.24	169
5 on BC exam	3.66	152
Placed via	Average grade in Calculus III	N
Passed Calculus II	2.74	9090
3 on BC exam	2.93	279
4 on BC exam	2.88	302
5 on BC exam	3.38	811

half of the exam score is based on six free response questions that must be hand scored or "read." This requires the participation of over a thousand "readers" who are brought together for 1 week each year. It is always arranged that roughly half are college or university faculty and half are high school teachers, ensuring a rich interchange of perspectives.

For many years following the publication of the MAA/NCTM letter in 1986, the College Board included this letter in the introduction to its course materials. The spirit of this letter is still found in its introduction (College Board, 2019b).

> "AP Calculus AB is designed to be the equivalent of a first semester college calculus course devoted to topics in differential and integral calculus. AP Calculus BC is designed to be the equivalent to both first and second semester college calculus courses.
> "Before studying calculus, all students should complete the equivalent of four years of secondary mathematics designed for college-bound students: courses that should prepare them with a strong foundation in reasoning with algebraic symbols and working with algebraic structures. Prospective calculus students should take courses in which they study algebra, geometry, trigonometry, analytic geometry, and elementary functions."

The College Board is sensitive to allegations that AP Calculus creates pressures to shortchange the normal preparation for university mathematics or that it offers a version of calculus that is inferior to what is taught in college. Yet beyond reviews of the intended course syllabi and what is required within its examinations, it has not been possible for the College Board to police either of its injunctions.

5 Validity of the AP calculus exams

Over the years, the College Board has sponsored several studies, summarized in what follows, into the validity of using its exams for advanced placement. We only look at their results for Calculus.

5.1 Morgan & Ramist, 1998

This was a large-scale study conducted in Fall 1991 at 21 colleges and universities chosen from among those that receive the greatest number of AP Calculus scores. It looked at students who received at least a 3 on an AP Calculus exam and chose to use this credit to skip at least one calculus class. It showed that even for students who scored a 3 on the AB Calculus exam, they did better in Calculus II then the average student who had passed Calculus I taken at that university (Table 1). The study suffered from several flaws: All that is reported are averages taken across all of the universities; there is no attempt to compare students with a given AP score with students who received a particular grade in Calculus I; and there is no attempt to control for the possibility that the population of students who earn AP credit and are sufficiently confident to skip Calculus I are not completely comparable to the population of those who take and pass Calculus I.

Note that the average grade in Calculus II for those who passed Calculus I is slightly different because not all universities could be used for the grade in Calculus III. The fact that a 4 on the BC exam predicts a lower score on Calculus III than a 3 is almost certainly a result of the fact that many universities, especially those with the most demanding courses, do not allow a student with a 3 on the BC exam to place directly into Calculus III. Among those that do, many

Table 2 Comparison of students who used AP exam grade to skip a course versus students who took and passed that course. Source: Morgan & Klaric, 2007

Placed via	Average grade in Calculus II	SAT adjusted grade
Passed Calculus I	2.43	
3 on AB exam	2.69	2.64
4 on AB exam	2.90	2.78
5 on AB exam	3.34	3.15
Placed via	**Average grade in Calculus III**	**SAT adjusted grade**
Passed Calculus II	2.50	
3 on BC exam	3.00	2.92
4 on BC exam	3.45	3.35
5 on BC exam	3.46	3.27

N = 5932 for Calc AB, N = 5411 for Calc BC

students with a 3 on the BC exam—especially those who are not confident of their ability—will choose not to place directly into Calculus III.

5.2 Morgan & Klaric, 2007

This was a large-scale study conducted in Fall 1994 at 22 colleges and universities chosen from among those that receive the greatest number of AP Calculus scores. The significant advantage over the previous study was that it adjusted the grades of those who took advantage of advanced placement, weighting their scores so that the distribution of SAT scores was comparable to that of students who had taken the previous course at that institution (Table 2).

5.3 Dodd et al., 2002

This study was conducted at the University of Texas, Austin, over a 5-year period: 1996–99. It looked at all of the students who used AP credit from the AB Calculus exam to place into Calculus II (M408D) and compared these to a stratified random sample of students in Calculus II who had passed Calculus I (M408C), stratifying the sample so that the SAT scores of the two groups were comparable. The average Calculus II grade of the AP students was 2.98. The average grade for students from the sample was 2.55.

5.4 Keng & Dodd, 2008

This study at the University of Texas, Austin, 1998–2001, compared students who had used AP credit to place into Calculus II (M408D) with four other groups: those who took an AP Calculus course but did not score a 3 or higher on the exam, those who did score a 3 or higher but chose to retake Calculus I, those who earned credit for Calculus I via dual enrollment, and those who had passed Calculus I (M408C) at UT-Austin. Similar to the previous study by Dodd, students in the last group were chosen via stratified random sample so that their SAT distribution matched that of the students who had used AP credit to place into Calculus II. Because this mainstream calculus sequence proceeds at a brisk pace, spending the second semester on sequences, series, and topics in multivariable calculus, students who brought credit from dual enrollment programs were only counted if they had passed courses covering both differential and integral single variable calculus (Table 3).

The lack of significance comparing (c) and (f) is a result of very few students in category (f). It would have more useful to compare (c) and (e) if the distribution of SAT scores were comparable, but the numbers were too small to allow for that.

5.5 Patterson & Ewing, 2013

This study was based on data from Fall, 2006 collected from 53 four-year colleges and universities. It compared students who had taken the AP exam and chose to skip the comparable course, in the case of Calculus, either Calculus I or Calculus II, and gone directly into a subsequent course with those who had taken the prerequisite course at their institution. It drew on 3468 students who had taken Calculus AB and 1574 students who had taken Calculus BC. This study conducted a more extensive propensity matching based on gender, racial/ethnic identity, anticipated major, Preliminary SAT (PSAT) scores, and high school grade point average. Matched students who used their score on the Calculus AB

Table 3 Comparison of grades (4 = A, 3 = B, 2 = C) in Calculus II by preparation for this course. Source: Keng & Dodd, 2008

Preparation for Calculus II (M408D)	Average grade	N
(a) 3 or higher on BC exam	3.43	1607
(b) Took Calculus I, SAT distribution matches 3 + on BC	3.16	1607
(c) 3 or higher on AB exam	3.13	3013
(d) Took Calculus I, SAT distribution matches 3 + on AB	3.03	3013
(e) 3 + on AB exam and took Calculus I	2.96	588
(f) Dual enrollment credit	2.93	339
(g) BC course but no credit for exam, took Calc I	2.82	496
(h) AB course but no credit for exam, took Calc I	2.45	1622

The following differences were significant at the 95% confidence level: (a) over (b), all 4 years; (c) over (d), 2 of 4 years; (c) over (e), 1 of 4 years; (a) over (f), 1 of 4 years; (a) over (g), all 4 years; (b) over (h), all 4 years. Differences were always significant when comparing those who did with those who did not pass the AP exam

or BC exam for advanced placement showed higher grades in the subsequent course than did those students who had taken the prerequisite course. The effect sizes were small, 0.192 for Calculus AB and 0.229 for Calculus BC, but in both cases, the students using advanced placement had 95% confidence intervals that were disjoint from and above the 95% confidence intervals of those who had taken the prerequisite course.

The more recent studies, especially that of Patterson and Ewing, have used some form of matching to avoid the trap of the earlier studies. But even the variables chosen for propensity matching are not the whole story. We know from the 2010 MAA survey (Bressoud, 2015) that 40% of university students in Calculus I were entitled to enroll directly in Calculus II, but chose not to do so. A 2002 study (Christman Morgan, 2002) found that of those who received a 5 on the Calculus AB exam, 77% took advantage of advanced placement, with a 4 it dropped to 64%, and for a 3 it was 39%. Choosing to take advantage of advanced placement thus reflects higher scores and almost certainly a higher degree of self-confidence.

A combination of ceiling effects and the role of self-confidence may account for the perceived benefit of going directly into the more advanced course. Nevertheless, there is nothing in any of these studies to suggest that the AP course is inferior to college Calculus I.

Of course, there are differences. The high school Calculus AB course is usually spread over an entire year, around 160 class meetings. A typical university Calculus I will meet only 40 times, perhaps with additional recitation sections. High school calculus is taught in smaller classes, with more uniform expectations for what students know when they enter the class, and with greater interaction with the teacher. Calculus in high school is a terminal course. In college, it is considered preparation for more advanced mathematics. This should set different expectations. However, an analysis of Calculus I final exams collected from 150 randomly selected colleges and universities in 2010 found that over 85% of questions "could be solved by simply retrieving rote knowledge from memory, or recalling and applying a procedure, requiring no understanding of an idea or why a procedure is valid" (Tallman et al., 2016, p. 120).

The College Board regularly surveys the leading colleges and universities in the United States to ensure that its mix of topics and emphases match common practice. In addition, the College Board periodically recruits universities that receive a large number of these exam scores to include a subset of the AP Exam questions within their own final exams, thus ensuring that the AP scores, 1 through 5, correspond to college grades, roughly 5 = A, 4 = B, 3 = C (Ewing et al., 2010).

6 Foundation building versus early calculus

Would high school students be better served with more time spent on algebra and precalculus rather than studying calculus? As part of the preparation for their 2009–2010 study, *Factors Influencing College Success in Mathematics* (FICSMath), Sadler, Sonnert and their team questioned high school mathematics teachers and university professors to determine their views on what best prepares students for success in college calculus (Wade et al., 2016). While there were many points of overlap, the differences were instructive. High school teachers emphasized classroom environment and use of real world examples, combined with a belief that calculus in high school was good preparation for university mathematics. University professors focused on the need for better skills in algebra and precalculus.

Representative of their responses, a professor exclaimed, "Stop wasting time on calculus in high school! Concentrate instead on making sure your graduating students have a rock-solid foundation." A high school teacher expressed the belief that "'having previously taken calculus while in high school' (regardless of the high school grade received) will be a stronger predictor of doing well in College Calculus than

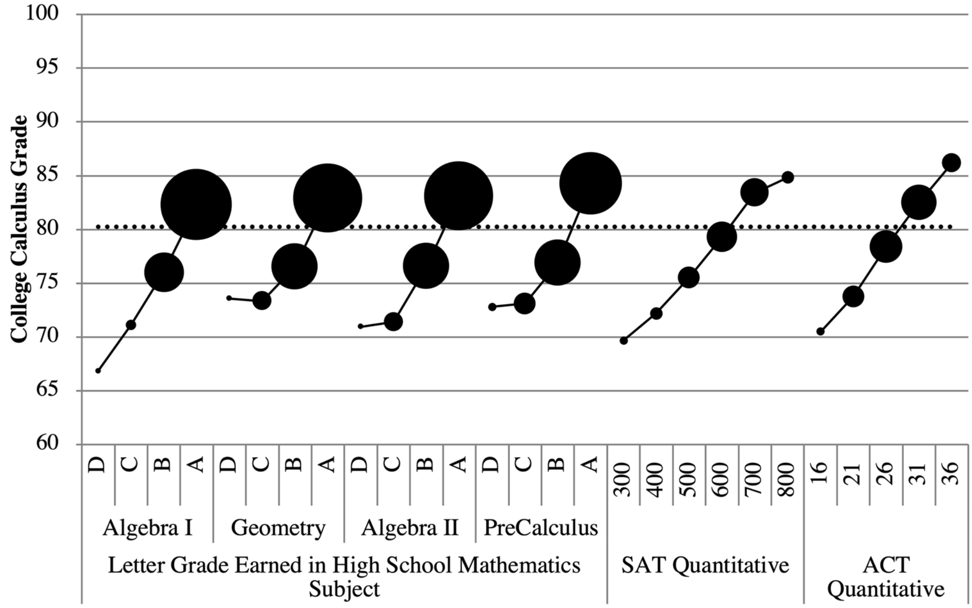

Fig. 5 Relationship between grade earned in college calculus and course grade or SAT/ACT score. The symbol area is proportional to the number of students in each group. The dotted line represents the mean grade (80.7). Source: Sadler and Sonnert, 2018, page 312

any activity, intervention, or reform" (Sadler and Sonnert, 2018, p. 292).

In response to these divergent opinions, one of the central research questions of FICSMath was.

> Which is the best preparation for college calculus—(a) a high level of mastery of mathematics considered preparatory for calculus (algebra, geometry, precalculus) or (b) taking calculus itself in high school? (Sadler and Sonnert, 2018)

As Sadler and Sonnert explain, these opinions reflect the difference between a belief in mathematics instruction that is strictly hierarchical—a given topic cannot be learned until all of the preliminary material is mastered—and a spiral approach that does not expect mastery on first contact but that anticipates returning to the material to deepen understanding.

To answer their research question, the study analyzed data collected from 6207 students in first semester university calculus, representing 216 professors at 133 randomly selected U.S. colleges and universities. What they found was truth on both sides. They collected data on a wide variety of factors that might influence performance in Calculus I, from high school experiences and intended career to race, gender, and parental education. Two factors dominated: preparation for calculus as measured by grades in high school mathematics courses and whether or not a student had taken calculus in high school. The relationship between grades and SAT/ACT scores and final grade in Calculus I is shown in Fig. 5.

A refined factor analysis led to the creation of a Calculus Preparation Composite (CPC), constructed from the four grades and the SAT/ACT scores. Normalized for a mean of 0 and a standard deviation of 1, they found that students with the same preparation score did better in Calculus I by about half a grade (5 points) if they had taken calculus in high school. The effect was most pronounced for weaker students and for those who took Calculus I in their first semester at university (Fig. 6). Perhaps most impressive is that for students whose preparation score was 1.5 standard deviations below the mean but had studied calculus in high school, the average grade in Calculus I was the same B minus as students at the mean score for preparation who had not studied calculus while in high school. This insight is not quite as surprising as it might seem since the CPC is based on courses taken before calculus. Thus, those who studied calculus in high school had an additional year of work with the mathematics needed in calculus. What is clear is that this year confers real benefits.

7 Equity issues

The College Board is aware of the problems of equity in the AP program. Comparing the distribution of AP Calculus exams that earned a 3 or higher (usually the cut-off for earning college credit), we see that over the 20 years from 1999 to 2019, the percentage of these exams taken by Black students has been constant at 2% (Figs. 7 and 8). This has been a driving force behind the development of a new suite of support materials, launched in 2019 and available as *AP Classroom*, consisting of unit guides, personal progress checks, a progress dashboard for teachers, and an extensive bank of AP questions (College Board, 2019c).

The most dramatic increases are among Asian and Hispanic students. Over this same period, the number of exams has almost tripled, from 158,000 in 1999 to 440,000 in 2019,

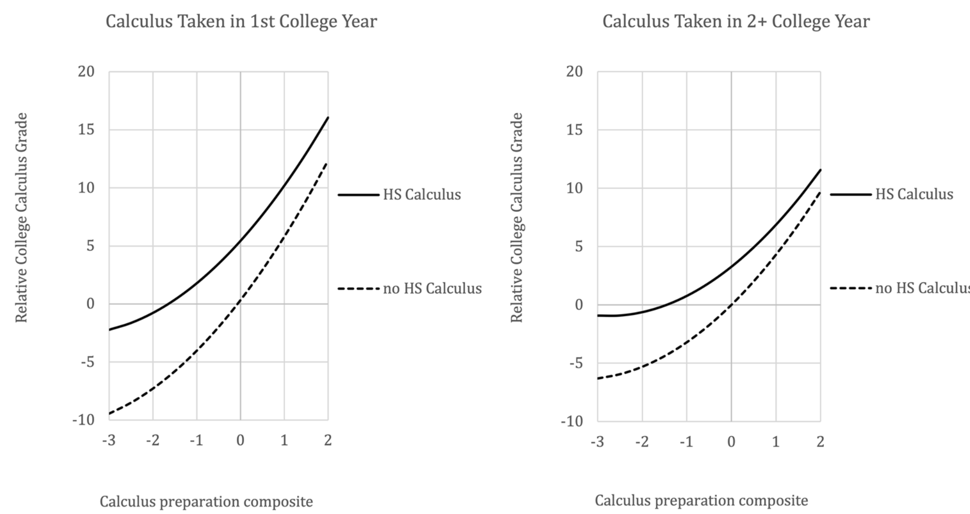

Fig. 6 Relationship between college calculus performance, high school preparation, taking high school calculus, and year taking calculus in college. Relative grade reflects points above or below the mean of 80.7. Source: Sadler and Sonnert 2018, page 319

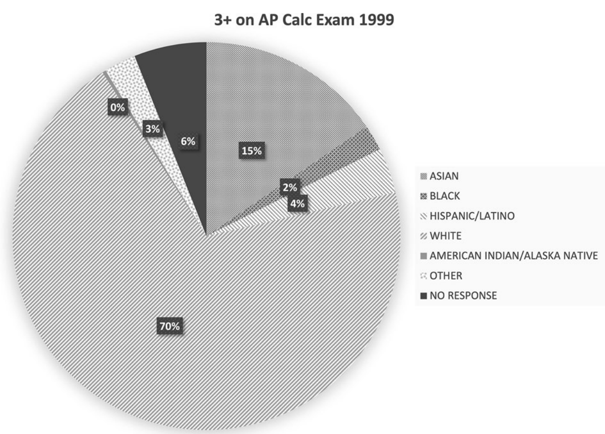

Fig. 7 Distribution by race/ethnicity of U.S. students who took either AP Calculus Exam in 1999 and scored a 3 or higher. Read clockwise from Asian = 15%. Actual percentage for American Indian/Alaska Native was 0.29%. Source: College Board, 1999

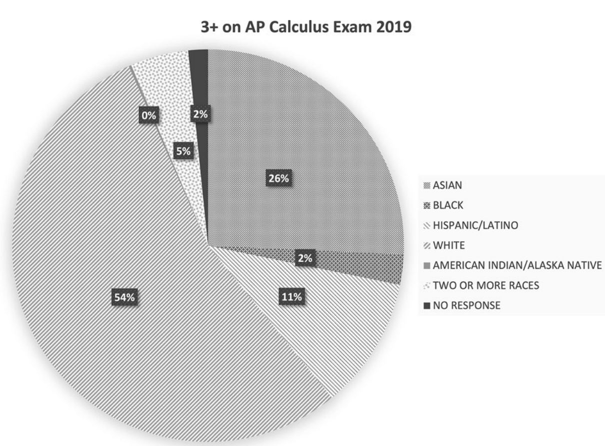

Fig. 8 Distribution by race/ethnicity of U.S. students who took either AP Calculus Exam in 2019 and scored a 3 or higher. Read clockwise from Asian = 26%. Actual percentage for American Indian/Alaska Native was 0.22%. Source: College Board, 2019a

Table 4 Percentage of students in each racial/ethnic category taking an AP calculus exam in the indicated year and percentage earning 3 or higher. Sources: College Board, 1999 and 2019a; NCES, 2019)

	White (%)	Black (%)	Hispanic (%)	Asian (%)
Take exam				
1999	5.7	1.6	2.6	21.2
2019	12.7	3.7	7.9	43.9
Earn ≥ 3				
1999	3.9	0.6	1.2	14.9
2019	8.6	1.5	3.5	33.7

Percentage calculated as number of students reporting that category on the AP calculus exam divided by the number of high school graduates in that category

so there still is improvement in terms of the percentage of students in each of the four largest racial/ethnic categories (Table 4).

Today, the dominant route toward Calculus I and a career in science, mathematics, or engineering involves an accelerated program that includes calculus in high school. In 2009, 50% of the students taking Calculus I had completed Algebra I at or before grade 8 (Sadler and Sonnert, 2018). In a 2010 survey at research universities, 78% of Calculus I students had finished Algebra II by the end of grade 10 (Bressoud, 2015).

But in order to take calculus in high school, a student needs to either start Algebra I by 8th grade or compress the subsequent courses. In the high school graduating class of 2013, 41% of those who had taken Algebra I in 8th grade or before went on to study calculus while in high school (Champion and Mesa 2017). Only 5% of those who waited until 9th grade to take Algebra I would go on to enroll in high school calculus.

Calculus in high school also requires access to this course. The 2018 National Survey of Science and Mathematics

Table 5 Percentage of students in the high school class of 2013 who completed a calculus course in high school by race/ethnicity, SES, score on the high school longitudinal study's mathematics exam administered in grade 9, and mathematics course taken in grade 9. Source: Champion & Mesa, 2017

	White (N = 8649) (%)	Black (N = 1511) (%)	Hispanic/Latino (N = 2307) (%)	Asian (N = 1229) (%)
Race/ethnicity	19	8	12	48
	Lowest quartile (%)	Middle quartiles (%)	Highest quartile (%)	
SES (N = 15,187)	7	15	38	
Math score grade 9 (N = 15,188)	2	11	50	
	Before Algebra I (N = 1788) (%)	Algebra I (N = 7249) (%)	Beyond Algebra I (N = 5703) (%)	
9th grade math course	2	5	41	

Quartiles refer to quartiles for the variable, either SES or Math score in grade 9

Education (Banilower et al. 2018) revealed that 48% of U.S. high schools do not offer calculus. While these are usually smaller schools and there often are other options such as an online course or enrollment in a local college, 17% of U.S. high school students do not have access to calculus (Banilower et al. 2018).

Table 5 illustrates the sharp discrepancies in the percentage of students who completed a calculus course in high school by race/ethnicity, socio-economic status (SES), mathematics score, and mathematics course taken in grade 9 (Champion and Mesa 2017).

Equity issues go much deeper than access to calculus in high school. Despite the number of students affected, pre-service training for secondary mathematics teachers seldom if ever includes preparation to teach calculus. Those who would teach this subject must seek their own in-service opportunities, requiring highly motivated teachers and financial support from their school.

Moreover, as documented in *Talking about Leaving Revisited* (Seymour and Hunter, 2019), the problems students encounter in high pressure university science and mathematics courses, especially Calculus, have much more to do with inadequate study skills and an inability to cope with the fast pace and impersonal nature of these courses than with any gaps in procedural knowledge. Schools in affluent districts are better placed to provide the potent challenges and experiences that equip students for the demands of university mathematics. Programs such as Emerging Scholars (Hsu et al. 2008) have been successful because they address these weaknesses.

8 Conclusion

Unintentionally, the United States has created a system in which the overwhelming majority of students heading toward careers in the physical, biological, or mathematical sciences or engineering take the same calculus course twice, once in the last year of high school and again in the first semester of university. The evidence suggests that this system is not completely dysfunctional since spiraling through this material can improve student performance during the second iteration. In addition, a few strong students are able to take advantage of high school calculus to move directly into more advanced mathematics. Unfortunately, few colleges or universities take advantage of this spiral. Because most colleges and universities still teach Calculus I to a mix of students who have and students who have not studied calculus before, they teach it as if none of their students had previously seen this material.

The situation that has evolved has created two significant downsides. The first is that it pushes all students toward calculus. Almost half the 800,000 who study calculus while in high school will not build on this course when they get to college. Many of them would be better served by a more intensive introduction to statistics, data science, and modeling.

The second downside is more disturbing. Because calculus in high school is now the expected preparation for calculus in college, the current system serves to widen inequities between those with access to well-resourced public schools and those without. The fact that most funding for public education in the U.S. is generated locally means that the obstacles faced by students of low SES are multiplied by the difficulties their schools face in offering the same courses available to those of high SES.

Restricting access to calculus in high school is unlikely to happen. School boards and principals will continue to value the cachet that comes with large enrollments in AP Calculus. Parents will want their children to have the advantage of this course in preparing for and getting into college. If there were evidence that offering calculus in high school is detrimental, that might make restrictions possible. But as reported in this article, what we know points in the other direction. The College Board has demonstrated that AP Calculus enhances the prospects of the strongest students. Sadler and Sonnert have shown that calculus in high school carries benefits, even for weaker students.

The best response is to strengthen the high school preparation of *all* students so that they experience challenging mathematics, mathematics that confronts them with unfamiliar problems and builds their ability to learn through reflective struggle. There is a growing awareness, exhibited in programs such as *Launch Years* (Charles A. Dana Center, 2020), that this does not need to be calculus or a course that prepares for calculus.

However, universities bear most of the responsibility for addressing the vast disparities among the students enrolled in calculus. Irrespective of the amount of calculus a student retains from a high school course, simple familiarity with the terminology and concepts create a huge gulf between those who have studied it before and those to whom it is new. We know from *Talking about Leaving Revisited* (Seymour and Hunter, 2019) that it can be incredibly discouraging to find oneself in a class where it seems that everyone else is familiar with the ideas and methods that are new to you. This is especially true for students whose sense of belonging in a mathematically intensive program is already fragile because of beliefs associated with their gender, race, ethnicity, or SES.

We also know that requiring students to repeat the precalculus they studied in high school is ineffective (Sonnert and Sadler, 2014; Thompson et al., 2007). Instead what is needed is a department-wide effort to understand the nature of the particular problems at that institution and to support promising interventions. Among the exemplars of what can be done are the University of Michigan (Carreon et al., 2018), San Diego State University (Apkarian et al., 2018), and the many universities in the SEMINAL program of the Association of Public and Land-Grant Universities (APLU, 2020).

Above all, we need to provide many more opportunities for constructive interaction between secondary and tertiary faculty. Each side of the high school to college divide faces a unique set of expectations, constraints, and challenges that the other often fails to appreciate. The transition from high school to college mathematics is successfully bridged only when all work together toward a common vision.

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