Background to the MAA/NCTM Statement on Calculus

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Evidence of a Problem

In 1984, the United States graduated 113,000 students with a bachelor's degree in engineering, the physical sciences, or the mathematical sciences. By 2010, 112,000 students graduated with a bachelor's degree in one of these disciplines. While there has been some variation over the past quarter century, the number of students earning degrees in these fields has stayed remarkably constant.

Nothing else has remained the same. On the supply side, more students are learning more mathematics in high school than ever before. In 1990, only 14% of high school graduates had completed precalculus or higher and 7% had taken a course in calculus. By 2009, the percentages were 35% and 17%, respectively.[4]

At the other end of the pipe is increased demand for engineers, scientists, mathematicians, and statisticians. In August 2011, President Obama's Council on Jobs and Competitiveness announced a new initiative to increase the number of engineering degrees earned each year by 10,000. As they headlined: "The U.S. Has a Shortage of Engineers, Hindering our Global Competitiveness and Threatening our Ability to Create and Keep High-Tech Jobs." [5] The President's Council of Advisers on Science and Technology (PCAST) published its report in February 2012, calling for one million additional Science, Technology, Engineering, and Mathematics (STEM) degrees over the next decade,[6] including in their evidence a report from the US Department of Commerce that projects a 17% increase in the need for graduates with STEM degrees over this period.[7]

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[4] Table 8 in NCES. *America's High School Graduates: Results of the 2009 NAEP High School Transcript Study*

[5] http://www.whitehouse.gov/the-press-office/2011/08/31/president-s-council-jobs-and-competitiveness-announces-industry-leaders-

[6] PCAST. *Report to the President. Engage to Excel.*

[7] US Dept. of Commerce. Economics and Statistics Administration. *STEM: Good Jobs Now and for the Future,* July 2011

What the members of the mathematical community—especially those in the Mathematical Association of America (MAA) and the National Council of Teachers of Mathematics (NCTM)—have known for a long time is that the pump that is pushing more students into more advanced mathematics ever earlier is not just ineffective: It is counter-productive. Too many students are moving too fast through preliminary courses so that they can get calculus onto their high school transcripts. The result is that even if they are able to pass high school calculus, they have established an inadequate foundation on which to build the mathematical knowledge required for a STEM career. Nothing demonstrates this more eloquently than the fact that from the high school class of 1992, one-third of those who took calculus in high school then enrolled in precalculus when they got to college,[8] and from the high school class of 2004, one in six of those who passed calculus in high school then took *remedial* mathematics in college.[9]

The rush to calculus has had another, more insidious, effect. Today, 61% of the students in college Calculus I have completed a calculus course while in secondary school. Of those, 61% earned an A in their high school class and 30% earned a 3 or higher on an Advanced Placement (AP) Calculus exam. Students who have not had the experience of a high quality calculus course in high school find themselves competing in Calculus I with those who have. At the extreme end of the dysfunctional system that has arisen, there are roughly 5000 students per year who study Calculus AB as juniors, Calculus BC as seniors, and then take Calculus I in their first year of college, essentially taking the same course in three successive years for full credit each time.[10]

Over the past decade a common and often self-fulfilling perception has blossomed among high school and college students that one cannot succeed in college calculus without having first done well in high school calculus. Many otherwise talented students give up on the prospect of a STEM career because they have not had access to a good calculus program in secondary school.

A History of Concern

The Advanced Placement Calculus program began in the 1950's as College Admission with Advanced Standing, an opportunity to provide challenging curricula to talented secondary school students by offering them college-level courses. Calculus made up a very small part of the early exams. The emphasis was on complex and unfamiliar problems that drew on all areas of mathematics. By the late 1950's, AP Calculus offered what was unmistakably a calculus exam. As the program grew, the emphasis shifted from assessing problem-solving ability to testing knowledge of calculus.

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8 NCES. National Education Longitudinal Study of 1988 (NELS:88). http://nces.ed.gov/surveys/nels88/

9 NCES. Education Longitudinal Study of 2002 (ELS:2002). http://nces.ed.gov/surveys/els2002/

10 All data in this paragraph are from the MAA study *Characteristics of Successful Programs in College Calculus*.

Beginning in the 1970s and greatly accelerating in the 1980s, district superintendants and chief state school officers began to see the AP program as a lever by which to raise secondary school quality by challenging the best students to engage in college-level work while still in high school. The hope was that the achievements of the best students would serve to inspire others and so raise the academic level of the entire school.

By 1982, over 200,000 students were studying calculus in high school.[11] Yet most of these classes constituted calculus instruction in name only, a reality reflected in the fact that only 15% of those who enrolled in these classes took the AP Calculus exam.

It was in this atmosphere that MAA directed its Committee on the Undergraduate Program in Mathematics (CUPM) to investigate what was happening in secondary school calculus and to make recommendations. The committee's report, published in 1987,[12] identified five problem areas:

1. Secondary school teacher qualifications and expectations.
2. Student qualifications and expectations.
3. The effect of repeating a course in college after having experienced success in a similar secondary school course.
4. College placement.
5. Lack of communication between secondary schools and colleges.

CUPM found that many students were short-changing their mathematical preparation so that they could enroll in calculus in the 12th grade, and too often this calculus course was nothing like the college course they thought they were experiencing. The committee made seventeen recommendations designed to address these problems. The two most important recommendations were elevated to a joint policy statement of MAA and NCTM, issued in 1986. The two societies recommended that

1. The calculus course offered in the 12th grade should be treated as a college-level course.
2. Students who enroll in a calculus course in secondary school should have demonstrated mastery of algebra, geometry, trigonometry, and coordinate geometry.

In some respects, the situation has improved considerably since this statement was issued. By 2009, over half of the students who studied calculus in secondary school took the AP Calculus exam. An aggressive program of professional development by College Board has reached a large number of teachers. The National Math and Science Initiative is working to introduce quality instruction in calculus to disadvantaged high schools.

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[11] CUPM Panel. 1987. Report of the CUPM Panel on Calculus Articulation: Problems in Transition from High School Calculus to College Calculus. *The American Mathematical Monthly*. Vol. 94, no. 8, 776–785.

[12] Ibid.

However, what had started as a means of engaging and challenging our most talented students has turned into the expected curriculum for those students who are heading to college and are able to keep up with an accelerated curriculum in mathematics. By the academic year 2011–12, over 650 thousand students were studying calculus in secondary school,[13] close to one-third of the two million students who graduate from secondary school and matriculate as full-time college students within a year.

There is now an expectation that every secondary school should offer AP Calculus or its equivalent, with the result that the demand for calculus teachers is outstripping the supply of those who are fully qualified. Within our schools, there is tremendous pressure to fill these classes, accelerating every student who might conceivably be ready for calculus by the senior year regardless of whether such a student might benefit from a slower and more thorough introduction to the traditional topics of high school mathematics. The AP Calculus program continues to grow at around 6% per year.

The problems that were observed in the 1987 report are still with us, now exacerbated by the massive scale of secondary school calculus today. Too many students are being accelerated, short-changing their preparation in and knowledge of algebra, geometry, trigonometry, and other precalculus topics. Too many students experience a secondary school calculus course that drills on the techniques and procedures that will enable them to successfully answer standard problems, but are never challenged to encounter and understand the conceptual foundations of calculus. Too many students arrive at college Calculus I and see a course that looks like a review of what they learned the year before. By the time they realize that the expectations of this course are very different from what they had previously experienced, it is often too late to get up to speed.

The Calculus Bottleneck

There is an even more fundamental problem that was recognized in the 1987 CUPM report. That is the assumption that college-level work in mathematics that is done in secondary school, especially for students heading into science or engineering, must be calculus. The result is that at least one course of calculus before graduation from secondary school has become the norm for the top quartile of college-bound students. It appears that most students who study calculus in secondary school do so not because of a desire to learn calculus, but because their peers are all on this track. The K-12 curriculum is directed toward getting those who are capable of doing so into calculus by grade 12.

This is reinforced by what happens on the other side of the secondary school to college transition.

In the 1960s, CUPM codified what is today the common undergraduate curriculum in mathematics for students heading into science or engineering. It is built upon the

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[13] From the NAEP high school transcript study of 2009, 53% of those who studied calculus in high school also took the AP Calculus exam. In 2012, 363,000 took the AP Calculus exam, suggesting that there were 680,000 students who studied calculus in high school.

assumption that students will study calculus in their first year and that all further mathematics courses may presume that the student has mastered single variable calculus. The unfortunate result is that at most colleges and universities today, a student who intends to major in science or engineering must complete single variable calculus before being allowed into any of the other required mathematics courses, the sole exception being statistics. Calculus has become the great bottleneck of mathematics.

It does not need to be so. The 1987 CUPM report recommended analytical geometry, discrete mathematics, and matrix algebra as alternatives to calculus for those who would study college-level mathematics in 12th grade. We also need to encourage alternatives to calculus for the first year of college. There is no reason for discrete mathematics or linear algebra or a course in transformational geometry to require calculus as a prerequisite.

More than this, there is real danger in funneling all of our potential science and engineering majors through a double dose of introductory calculus, once on each side of the transition from high school to college. For those students who, for whatever reason, have had a bad experience of calculus in secondary school, the prospect of repeating their experience can dissuade them from continuing their study of mathematics. Even for those who enjoyed their secondary school calculus, beginning college mathematics with a course that looks very much like what they mastered the previous year is at best uninspiring and at worst could lull them into believing that this second iteration of introductory calculus does not require significant effort.

It is no wonder that even well prepared students are put off by their experience of mathematics in college. The college curriculum should offer students an experience that is new and engaging, helping to open their understanding of the world of mathematics while strengthening their mastery of tools that they will need if they choose to pursue a mathematically intensive discipline.

The New Recommendations: A Vision of Secondary and Collegiate Collaboration

On both sides of the transition from secondary to college mathematics, college faculty and secondary teachers must work together to strengthen the mathematics curriculum so that those students who intend to pursue a mathematically intensive career can acquire the mathematical knowledge and capabilities needed for such a career. In March 2012, MAA and NCTM adopted a joint policy statement with the following three recommendations of which the first two reiterate those of 1986.

Recommendation 1: Students who enroll in a calculus course in secondary school should have demonstrated mastery of algebra, geometry, trigonometry, and coordinate geometry.

A common refrain of mathematics faculty in our colleges and universities is that students who fail calculus do so not because they have not been able to learn the calculus but because of a more fundamental lack of the skills and understandings of precalculus. Mastery of algebra, trigonometry, exponentials and logarithms, and an understanding of the role of function in linking co-varying quantities are all essential ingredients for the study of calculus. We need tools for assessing student readiness for calculus. We also need strong alternatives to calculus for our secondary schools. The student who is skilled in algebraic and geometric thinking is far better prepared for university-level mathematics than one who has memorized techniques for differentiation and integration.

Recommendation 2: The calculus course offered in secondary school should have the substance of a mainstream college-level course.

We now have evidence that simply having studied calculus in high school confers little or no advantage to students when they enter the first calculus course in college. Benefits do not appear until the student has learned the subject well enough to earn a 3 or higher on an AP Calculus exam.[14] We also know that students who have taken calculus in high school enter college with an inflated sense of their ability to handle university-level mathematics.[15] We need clear guidelines for what is meant by "college calculus in high school" and access to data on which calculus courses are or are not preparing students for university-level mathematics. This includes guidelines for dual enrollment programs, which often have been shown to provide inadequate preparation.[16]

Recommendation 3: The college curriculum should acknowledge the ubiquity of calculus in secondary school, shape the college calculus curriculum so that it is appropriate for those who have experienced introductory calculus in high school, and offer alternatives to calculus.

Colleges and universities can no longer pretend that they can teach calculus the same way that they did in 1990. First of all, the students who twenty years ago made up the top third of the students in Calculus I now skip this course when they enter college. Second, most students enter college calculus already familiar with many of the standard techniques and procedures and a strong preconception of what this course is about and what is required in order to succeed. They are primed to ignore the conceptual development of the subject. Third, the push to accelerate high school students means that many of these students enter college with a weaker mathematical foundation than they would have had a generation ago. And finally, revisiting material they have already studied is not an effective means of engaging students and building a desire to learn mathematics. There are many ways of

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[14] Phillip Sadler. 2012. *Factors Influencing STEM Preparedness: From Algebra to Calculus*. NCTM 2012 Research Presession.

[15] From the MAA study *Characteristics of Successful Programs in College Calculus*: Of the students in mainstream Calculus I who had studied calculus while in high school, over 60% received an A in their high school class and almost two-thirds expected to get an A for their college calculus class. In fact, only a quarter of the students in Calculus I who had taken calculus in high school received an A in the college class.

[16] See David Bressoud. *The Dangers of Dual Enrollment*. http://maa.org/columns/launchings/launchings_07_07.html

handling these problems. One is to make it clear that this is not their high school calculus class, either by taking a modeling approach that focuses on differential equations or going to the other extreme and making this an introductory course in analysis. One also can start students with a course that is not calculus, such as discrete mathematics or linear algebra.

Conclusion

The United States has fallen into a seriously dysfunctional system for preparing students for careers in science and engineering, guaranteeing that all but the very best rush through essential parts of the mathematics curriculum and then are forced to sit and spin their wheels while they try to compensate for what was missed. It will take time and work by all involved to repair the transition from high school to college. We cannot afford to wait.